

m_3^3 -Convex Geometries are A -free.

J. Cáceres, Dept. of Stats. and Applied Math.
Univ. of Almeria, 04120, Almeria Spain
e-mail: jcaceres@ual.es

O.R. Oellermann*, Dept. of Math. and Stats., Univ. of Winnipeg,
515 Portage Ave, Winnipeg, R3B 2E9, Canada
e-mail: o.oellermann@uwinnipeg.ca

M.L. Puertas, Dept. of Stats. and Applied Math.,
Univ. of Almeria, 04120, Almeria, Spain
e-mail: mpuertas@ual.es

July 7, 2011

Abstract

Let V be a finite set and \mathcal{M} a collection of subsets of V . Then \mathcal{M} is an alignment of V if and only if \mathcal{M} is closed under taking intersections and contains both V and the empty set. If \mathcal{M} is an alignment of V , then the elements of \mathcal{M} are called convex sets and the pair (V, \mathcal{M}) is called an aligned space. If $S \subseteq V$, then the convex hull of S is the smallest convex set that contains S . Suppose $X \in \mathcal{M}$. Then $x \in X$ is an extreme point for X if $X \setminus \{x\} \notin \mathcal{M}$. The collection of all extreme points of X is denoted by $ex(X)$. A convex geometry on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. Let $G = (V, E)$ be a connected graph and U a set of vertices of G . A subgraph T of G containing U is a minimal U -tree if T is a tree and if every vertex of $V(T) \setminus U$ is a cut-vertex of the subgraph induced by $V(T)$. The monophonic interval of U is the collection of all vertices of G that belong to some minimal U -tree. A set S of vertices in a graph is m_k -convex if it contains the monophonic interval of every k -set of vertices in S . A set of vertices S of a graph is m^3 -convex if for every pair u, v of vertices in S , the vertices on every induced path of length at least 3 are contained in S . A set S is m_3^3 -convex if it is both m_3 - and m^3 -convex. We show that if the m_3^3 -convex sets form a convex geometry, then G is A -free.

Key Words: minimal trees, monophonic intervals of sets, k -monophonic convexity, convex geometries

AMS subject classification: 05C75, 05C12, 05C17

*Supported by an NSERC grant CANADA.

1 Introduction

Let G and F be graphs. Then F is an *induced subgraph* of G if F is a subgraph of G and for every $u, v \in V(F)$, $uv \in E(F)$ if and only if $uv \in E(G)$. We say a graph G is F -free if it does not contain F as an induced subgraph. Suppose \mathcal{C} is a collection of graphs. Then G is \mathcal{C} -free if G is F -free for every $F \in \mathcal{C}$. If F is a path or cycle that is a subgraph of G , then F has a *chord* if it is not an induced subgraph of G , i.e., F has two vertices that are adjacent in G but not in F . An induced cycle of length at least 5 is called a *hole*.

Let V be a finite set and \mathcal{M} a collection of subsets of V . Then \mathcal{M} is an *alignment* of V if and only if \mathcal{M} is closed under taking intersections and contains both V and the empty set. If \mathcal{M} is an alignment of V , then the elements of \mathcal{M} are called *convex sets* and the pair (V, \mathcal{M}) is called an *aligned space*. If $S \subseteq V$, then the *convex hull* of S is the smallest convex set that contains S . Suppose $X \in \mathcal{M}$. Then $x \in X$ is an *extreme point* for X if $X \setminus \{x\} \in \mathcal{M}$. The collection of all extreme points of X is denoted by $ex(X)$. A *convex geometry* on a finite set V is an aligned space (V, \mathcal{M}) with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the *Minkowski-Krein-Milman* (MKM) property. For a more extensive overview of other abstract convex structures see [13]. Convexities associated with the vertex set of a graph are discussed for example in [3]. Their study is of interest in Computational Geometry and has applications in Game Theory [2].

Convexities on the vertex set of a graph are usually defined in terms of some type of ‘intervals’. Suppose G is a connected graph and u, v two vertices of G . Then a $u - v$ *geodesic* is a shortest $u - v$ path in G . Such geodesics are necessarily induced paths. However, not all induced paths are geodesics. The g -interval (respectively, m -interval) between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some $u - v$ geodesic (respectively, induced $u - v$ path) in G and is denoted by $I_g[u, v]$ (respectively, $I_m[u, v]$).

A subset S of vertices of a graph is said to be g -convex (m -convex) if it contains the g -interval (m -interval) between every pair of vertices in S . It is not difficult to see that the collection of all g -convex (m -convex) sets is an alignment of V . A vertex v is an extreme point for a g -convex (or m -convex) set S if and only if v is simplicial in the subgraph induced by S , i.e., every two neighbours of v in S are adjacent. Of course the convex hull of the extreme points of a convex set S is contained in S , but equality holds only in special cases. In [6] those graphs for which the g -convex sets form a convex geometry are characterized as the chordal 3-fan-free graphs (see Fig. 1). These are precisely the chordal, distance-hereditary graphs (see [1, 7]). In the same paper it is shown that the chordal graphs are precisely those graphs for which the m -convex sets form a convex geometry.

For what follows we use P_k to denote an induced path of order k . A vertex is simplicial in a set S of vertices if and only if it is not the centre vertex of an induced P_3 in $\langle S \rangle$. Jamison and Olariu [8] relaxed this condition. They defined a vertex to be *semisimplicial* in S if and only if it is not a centre vertex of an induced P_4 in $\langle S \rangle$.

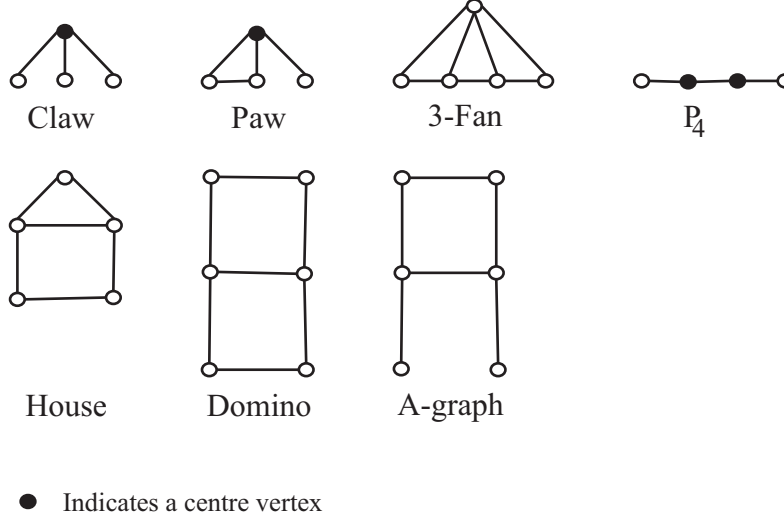


Figure 1: Special Graphs

Dragan, Nicolai and Brandstädt [5] introduced another convexity notion that relies on induced paths. The m^3 -interval between a pair u, v of vertices in a graph G , denoted by $I_{m^3}[u, v]$, is the collection of all vertices of G that belong to an induced $u - v$ path of length at least 3. Let G be a graph with vertex set V . A set $S \subseteq V$ is m^3 -convex if and only if for every pair u, v of vertices of S the vertices of the m^3 -interval between u and v belong to S . As in the other cases the collection of all m^3 -convex sets is an alignment. Note that an m^3 -convex set is not necessarily connected. It is shown in [5] that the extreme points of an m^3 -convex set are precisely the semisimplicial vertices of $\langle S \rangle$. Moreover, those graphs for which the m^3 -convex sets form a convex geometry are characterized in [5] as the (house, hole, domino, A)-free graphs (see Fig. 1).

More recently a graph convexity that generalizes g -convexity was introduced (see [11]). The *Steiner interval* of a set S of vertices in a connected graph G , denoted by $I(S)$, is the union of all vertices of G that lie on some *Steiner tree* for S , i.e., a connected subgraph that contains S and has the minimum number of edges among all such subgraphs. Steiner intervals have been studied for example in [9, 12]. A set S of vertices in a graph G is k -Steiner convex (g_k -convex) if the Steiner interval of every collection of k vertices of S is contained in S . Thus S is g_2 -convex if and only if it is g -convex. The collection of g_k -convex sets forms an aligned space. We call an extreme point of a g_k -convex set a k -Steiner simplicial vertex, abbreviated kSS vertex.

The extreme points of g_3 -convex sets S , i.e., the $3SS$ vertices are characterized in [4] as those vertices that are **not** a centre vertex of an induced claw, paw or P_4 , in $\langle S \rangle$ see Fig. 1. Thus a $3SS$ vertex is semisimplicial. Apart from the g_k -convexity, for a fixed k , other graph convexities that (i) depend on more than one value of k and (ii) combine the g_3 convexity and the geodesic coun-

terpart of the m^3 -convexity were introduced and studied in [10]. In particular characterizations of convex geometries for several of these graph convexities are given.

The notion of an induced path between a pair of vertices can be extended to three or more vertices. This gives rise to graph convexities that extend the m -convexity. Let U be a set of at least two vertices in a connected graph G . A subgraph H containing U is a *minimal U -tree* if H is a tree and if every vertex $v \in V(H) \setminus U$ is a cut-vertex of $\langle V(H) \rangle$. Thus if $U = \{u, v\}$, then a minimal U -tree is just an induced $u - v$ path. Moreover, every Steiner tree for a set U of vertices is a minimal U -tree. The collection of all vertices that belong to some minimal U -tree is called the *monophonic interval of U* and is denoted by $I_m(U)$. A set S of vertices is *k -monophonic convex*, abbreviated as m_k -convex, if it contains the monophonic interval of every subset U of k vertices of S . Thus a set of vertices in G is a monophonic convex set if and only if it is a m_2 -convex set. By combining the m_3 -convexity with the m^3 -convexity introduced in [5], we obtain a graph convexity that extends the graph convexity studied in [10]. More specifically we define a set S of vertices in a connected graph to be *m_3^3 -convex* if S is both m^3 - and m_3 -convex. In this paper we show that if the m_3^3 -convex alignment forms a convex geometry then G is A -free. We use the fact that these graphs are F -free for several other graphs F . In particular G is easily seen to be house, hole, and domino free. Moreover the graphs of Fig. 2 are forbidden. A graph G is a *replicated twin C_4* if it is isomorphic to any one of the four graphs shown in Fig. 2(a), where any subset of the dashed edges may belong to G . The collection of the four replicated twin C_4 graphs is denoted by \mathcal{R}_{C_4} . A graph F is a *tailed twin C_4* if it is isomorphic to one of the two graphs shown in Fig. 2(b) where again any subset of the dotted edges may be chosen to belong to F . We denote the collection of tailed twin C_4 's by \mathcal{T}_{C_4} .

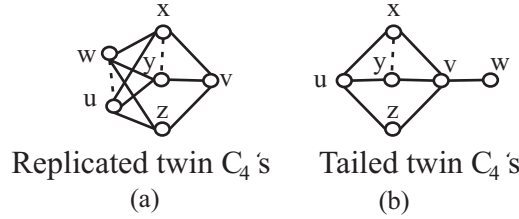


Figure 2: Forbidden subgraphs for m_3^3 -convex geometries

2 m_3^3 -Convex Geometries are A -Free

Recall that the graphs for which the m^3 -convex sets form a convex geometry are characterized in [5] as the (house, hole, domino, A)-free graphs. The proof of this characterization depends on the following useful result also proven in [5]:

Theorem 1. *If G is a (house, hole, domino, A)-free graph, then every vertex of G is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices.*

In [5] several ‘local’ convexities related to the m^3 -convexity were studied. For a set S of vertices in a graph G , $N[S]$ is $S \cup N(S)$ where $N(S)$ is the collection of all vertices adjacent with some vertex of S . A set S of vertices in a graph is connected if $\langle S \rangle$ is connected. The following useful result was established in [5].

Theorem 2. *A graph G is (house, hole, domino)-free if and only if $N[S]$ is m^3 -convex for all connected sets S of vertices of G .*

Theorem 3. *If $G = (V, E)$ is a graph such that $(V, \mathcal{M}_{m^3}(G))$ is a convex geometry, then G is A -free.*

Proof. Observe first that G is (house, hole, domino, \mathcal{R}_{C_4} , \mathcal{T}_{C_4})-free. Suppose F is a house, hole, domino, replicated twin C_4 or a tailed twin C_4 . Then F has at most one $3SS$ vertex. Suppose G is a graph that contains F as an induced subgraph. Then the set of extreme points of the convex hull of $V(F)$ is contained in the collection of $3SS$ vertices of F . So the convex hull of the extreme points of the m^3 -convex hull of $V(F)$ is empty or consists of a single vertex. So in this case the m^3 -convex alignment of G does not form a convex geometry.

If S is a set of vertices of a graph G , then $I_{m^3}(S) = \cup \{I_{m^3}[x, y] | x, y \in S\}$.

To show that G contains no A as an induced subgraph we prove a series of lemmas.

Lemma 1. *Suppose $G = (V, E)$ is a graph for which $(V, \mathcal{M}_{m^3}(G))$ is a convex geometry. Then for every $a, b \in V$, $I_{m^3}(I_m[a, b]) \subseteq I_m[a, b]$.*

Proof. By the above observation G is (house, hole, domino, \mathcal{R}_{C_4} , \mathcal{T}_{C_4})-free. If $ab \in E$ then $I_{m^3}(I_m[a, b]) \subseteq I_m[a, b] = \{a, b\}$. So we may assume $ab \notin E$. If $I_{m^3}(I_m[a, b]) \not\subseteq I_m[a, b]$, there is a vertex $w \notin I_m[a, b]$ that lies on an induced path between two vertices of $I_m[a, b]$. Among all such induced paths of length at least 3 containing w , let Q be one with a minimum number of edges. Suppose Q is a $u - v$ path. Clearly $\{u, v\} \neq \{a, b\}$; otherwise, $w \in I_m[a, b]$. Let $Q : (u =) v_1 v_2 \dots v_k (= v)$. (Suppose $w = v_i$.) Then w is not adjacent with two non-adjacent vertices of any induced $a - b$ path; otherwise, w lies on an induced $a - b$ path.

Case 1 Suppose u and v lie on a common induced $a - b$ path P . We may assume u precedes v on such a path. Moreover, we may assume that all internal vertices of Q are not on P . For if $v_j \in V(P)$, $1 < j < k$, then either $Q[v_1, v_j]$ or $Q[v_j, v_k]$ contains w , say the former. Since Q is an induced path, so is $Q[v_1, v_j]$. Hence $v_1 v_j \notin E$. Thus $Q[v_1, v_j]$ must have length at least 3; otherwise w is adjacent with a pair of nonadjacent vertices of P , implying that G contains an induced $a - b$ path passing through w , contrary to assumption. But then we have a contradiction to our choice of Q .

Let $S_1 = P[u, v] \setminus \{u, v\}$ and $S_2 = Q[u, v] \setminus \{u, v\}$. Then $\langle S_i \rangle$ is connected for $i = 1, 2$. By Theorem 2, $N[S_i]$ is m^3 -convex. Since u and v both belong to $N[S_i]$, every vertex of Q must be adjacent with an internal vertex of $P[u, v]$. This is true in particular for w . Since $P[a, u]$ followed by Q and then $P[v, b]$ is an $a - b$ path that contains w it cannot be induced. Some vertex of $P[a, u] \setminus \{u\}$ or a vertex of $P[v, b] \setminus \{v\}$ must be adjacent with an internal vertex of Q ; say the former occurs. Let x be the first vertex of $P[a, u]$ that is adjacent with an internal vertex y of Q . Let r be the first vertex on $Q[y, v]$ that is adjacent with a vertex of $P[v, b]$ (possibly r is v_{k-1}). Let s be the last vertex of $P[v, b]$ adjacent with r . Then the path $H : P[a, x]xyQ[y, r]rsP[s, b]$ is an induced $a - b$ path and thus does not contain w . So w is an internal vertex of $Q[u, y]$ or of $Q[r, v]$; suppose the former. Since H is connected, $N[V(H)]$ is m^3 -convex by Theorem 2. Since $a, b \in N[V(H)]$ and as P has length at least 3, $N[V(H)]$ must contain every vertex of P . Thus $I_{m^3}[u, v] \subseteq N[V(H)]$. Hence w is adjacent with a vertex of H . Since w is adjacent with an internal vertex of $P[u, v]$, w is not adjacent with any vertex of $P[a, x]$ nor $P[s, b]$. Since Q is an induced path, the only vertex of H to which w can be adjacent is y . So y follows w on Q . Since u and y belong to $I_m[a, b]$ and as $Q[u, y]$ is an induced path containing w , it follows that w must be adjacent with u ; otherwise, we have a contradiction to our choice of Q . Let x' be the last vertex on $P[x, u]$ to which y is adjacent. Then $x'u \in E$; otherwise $P[x', u]uwyx'$ is an induced cycle of length at least 5. Let z be the first internal vertex of $P[u, v]$ to which w is adjacent. (By an earlier observation z exists.) Then $uz \in E$; otherwise, w lies on an induced $a - b$ path. Also $yz \in E$; otherwise, $\langle \{x', u, w, y, z\} \rangle$ is a house. If $r \neq y$, let y' be the neighbour of y on $Q[y, r]$. Then $u, y' \in I_m[a, b]$ and $Q[u, y']$ is an induced path between two vertices of $I_m[a, b]$ having length 3 and containing w , contrary to our choice of Q . So $r = y$. So $P[x', s]syx'$ is a cycle of length at least 5. Since $yu \notin E$, $x'uzyx'$ is an induced 4-cycle. Let z' be the first vertex after z on $P[z, s]$ to which y is adjacent (perhaps $z' = s$). Then $P[z, z']z'yz$ is an induced cycle and hence has length 3 or 4. This cycle together with the 4-cycles $x'yzux'$ produces either a house or a domino both of which are forbidden. So we may assume that Q is an induced $u - v$ path between vertices u and v of $I_m[a, b]$ that do not belong to the same induced $a - b$ path. Indeed we may assume if u and v are any non-adjacent vertices that lie on the same induced $a - b$ path, then $I_{m^3}[u, v] \subseteq I_m[a, b]$.

Case 2 Suppose u and v lie on two internally disjoint $a - b$ paths P_u and P_v , respectively. We may assume $\{u, v\} \cap \{a, b\} = \emptyset$; otherwise, we are in Case 1.

We show first that no internal vertex of Q belongs to P_u or P_v . Suppose some internal vertex of $Q[u, w]$ or $Q[w, v]$, say $Q[u, w]$ belongs to P_u or P_v . However, no internal vertex of $Q[u, w]$ belongs to P_v ; otherwise, either the situation arises that was considered in Case 1 or there is an induced $a - b$ path containing w . So we may assume that an internal vertex of $Q[u, w]$ lies on P_u . Let u' be the last such vertex. Then $Q[u', v]$ contains w and is an induced path between two vertices of $I_m[a, b]$ that is shorter than Q . So $Q[u', v]$ has length 2; otherwise we have a contradiction to our choice of Q . So $Q[u', v]$ must be the path $u'wv$. Since Q has length at least 3 and by our choice of Q one of the neighbours of

u' on P_u must be u . So one of the configurations shown in Fig. 3 must occur where solid lines are edges and dashed lines represent subpaths of P_u and P_v . We may assume that the configuration in (a) occurs. The argument for the configuration in (b) is similar.

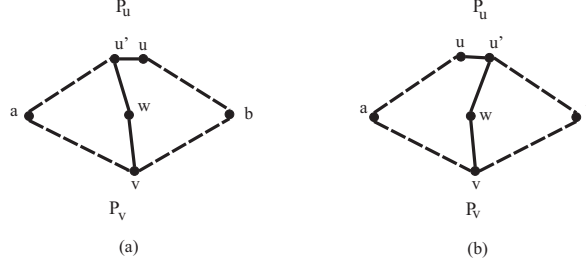


Figure 3: Two configurations in Case 2

Since Q is induced, v is not adjacent to u' or u and w is not adjacent with u . Let v_L and v_R be the neighbours of v on $P_v[a, v]$ and $P_v[v, b]$, respectively. If u' is adjacent with a vertex r of $P_v[v_R, b] - v_R$ then $ru'wv$ is an induced path of length 3 containing w and whose end vertices lie on the same induced $a - b$ path. By Case 1, this situation cannot occur. So the only vertex of $P_v[v_R, b]$ to which u' can be adjacent is v_R . Similarly, the only vertex of $P_v[a, v_L]$ to which u' can be adjacent is v_L . Using a similar argument and the fact that $vu \notin E$, we see that v is not adjacent with any vertex of $P_u[u, b]$. Moreover, w is not adjacent with any vertex of $P_u[u, b]$; otherwise, w lies on an induced $a - b$ path. The path obtained by taking $P_v[a, v]$ followed by vwu' and then $P_u[u', b]$ is an $a - b$ path that contains w . Hence this path is not induced. Suppose first that $wv_L \notin E$. So some vertex of $P_v[a, v]$ is adjacent with some vertex of $P_u[u', b]$. Since v is not adjacent with any vertex of $P_u[u', b]$, some vertex of $P_v[a, v_L]$ is adjacent with some vertex of $P_u[u', b]$. Let z be a vertex closest to v on $P_v[a, v]$ that is adjacent with a vertex of $P_u[u', b]$ and let y be such a neighbour of z closest to u' on $P_u[u', b]$. Observe that $y = u'$ and $z = v_L$; otherwise, the cycle $P_v[z, v]vwu'P_u[u'y]yz$ is an induced cycle of length at least 5. Let x be the vertex closest to u' on $P_u[u', b]$ that is adjacent with a vertex of $P_v[v, b]$ (possibly $x = b$). Let x' be the neighbour of x on $P_v[v, b]$ closest to v . By the above observation $x' \neq v$. The cycle $P_u[u', x]xx'P_v[x', v]vwu'$ is induced and has length at least 5 unless $x = u'$ and $x' = v_R$. So u' is adjacent with both v_L and v_R . Observe that u is either adjacent with both v_L and v_R or neither of these two vertices; otherwise, $\langle \{v_L, v, v_R, u', u\} \rangle$ is a house. We show next that no vertex of $P_u[u, b]$ is adjacent with v_L . Suppose r is a vertex on $P_u[u, b]$ closest to u that is adjacent with v_L . First observe that $r \neq u$ for if $uv_L \in E$, then $\langle \{u, u', w, v, v_L\} \rangle$ is a house. So r must be the neighbour of u on $P_u[u, b]$; otherwise, G has a hole. However then $\langle \{u', u, r, v_L, v, w\} \rangle$ is a domino. So v_L is not adjacent with any vertex of $P_u[u, b]$. Let $C : v_Ru'P_u[u', b]P_v[b, v_R]$. Then C is a cycle of length at least 5 and hence has chords. Now u' is not adjacent with any vertex of $P_v[v_R, b]$ other than v_R ; otherwise, w lies on an

induced path of length 3 between two vertices of $I_m[a, b]$ that belong to the same induced $a - b$ path, a case already dealt with. Since $uv_L \notin E$, $uv_R \notin E$. Suppose u is adjacent with an internal vertex of $P_v[v_R, b]$. Let s be such a vertex closest to v_R . So $s \neq v_R$. Since G contains no holes, s is adjacent with v_R . But then $\langle \{u', u, s, v_R, v, v_L\} \rangle$ is a domino. So the neighbour r of u on $P_u[u, b]$ is incident with a chord of C . Since G has no holes $rv_R \in E$. But then $\langle \{u', u, r, v_R, v, v_L\} \rangle$ is a domino. Suppose now that $wv_L \in E$. Then $wv_R \notin E$. Let $C' : P_u[u', b]P_v[b, v]vwu'$. Then C' is a cycle of length at least 5 and hence has no chords. Since neither w nor v are incident with chords of C' , $u'v_R \in E$. If $uv_R \in E$ $\langle \{u, u', v_R, v, w\} \rangle$ is a house. Note that u' is not adjacent with an internal vertex of $P_v[v_R, b]$; otherwise, if t is such a neighbour of u' , then $tu'vw$ is an induced path of length 3 between two vertices of $I_m[a, b]$ that lie on the same induced $a - b$ path, a case already considered. Let r be the neighbour of u on $P_u[u, b]$ and s the neighbour of v_R on $P_v[v_R, b]$. Then either v_Rr or us is an edge; otherwise, G has a hole. But then $\langle \{u', u, r, v_R, v, w\} \rangle$ or $\langle \{u', u, s, v_R, v, w\} \rangle$ is a domino. So no internal vertex of Q belongs to P_u or to P_v .

Let $Q : (u =)v_1v_2 \dots v_k(= v)$. Let u_L and u_R be the neighbours of u on $P_u[a, u]$ and $P_u[u, b]$, respectively and v_L and v_R the neighbours of v on $P_v[a, v]$ and $P_v[v, b]$, respectively. Let $S_1 = V(P_u[u_R, b]) \cup V(P_v[b, v_R])$ and $S_2 = V(P_u[a, u_L]) \cup V(P_v[a, v_L])$. Since $\langle S_i \rangle$ is connected for $i = 1, 2$, it follows from Theorem 2 that $N[S_i]$ is m^3 -convex. Since $u, v \in N[S_i]$ for $i = 1, 2$, every vertex of Q is adjacent with a vertex of S_i for $i = 1, 2$. In particular w is adjacent with a vertex of S_i for $i = 1, 2$. However, w is not adjacent with a pair of nonadjacent vertices of P_u nor a pair of nonadjacent vertices of P_v . So without loss of generality we may assume that w is adjacent with a vertex of $P_v[v_R, b]$ and a vertex of $P_u[a, u_L]$. Also w is not adjacent with either a or b ; otherwise, w lies on an induced $a - b$ path.

If v_2 is adjacent with two non-adjacent vertices of P_u (or if v_{k-1} is adjacent with two nonadjacent vertices of P_v), then $v_2 \neq w$ (and $v_{k-1} \neq w$, respectively) and $Q[v_2, v]$ (or $Q[u, v_{k-1}]$, respectively) is an induced path between two vertices of $I_m[a, b]$ that is shorter than Q and contains w . By our choice of Q this can only happen if Q has length 3.

We consider two subcases that depend on the length of Q .

Subcase 2.1 Suppose Q has length 3.

Then v_2 or v_3 is w , say $v_3 = w$. The case where $v_2 = w$ can be argued similarly. From the above, we may assume that w is adjacent with an internal vertex of $P_v[v, b]$ and an internal vertex of $P_u[a, u]$. The only vertex of $P_v[v_R, b]$ that can be adjacent with w is v_R ; otherwise, w lies on an induced $a - b$ path. So $wv_R \in E$. Now it follows that w is not adjacent with a vertex of $P_v[a, v_L]$. Thus $\langle \{v_2, w, v, v_L, v_R\} \rangle$ is a house unless $v_2v_R \in E$. If $v_2v_R \in E(G)$, then $wv_L, wv_R \notin E$; otherwise, $\langle \{u, v_2, v_3, v, v_L\} \rangle$ or $\langle \{u, v_2, v_3, v, v_R\} \rangle$ is a house. So $\langle \{u, v_2, v_3, v_L, v, v_R\} \rangle$ is a tailed twin C_4 which is forbidden. So this subcase cannot occur.

Subcase 2.2 Suppose Q has length at least 4.

By an earlier observation, v_2 is not adjacent with a pair of non-adjacent vertices

of P_u and v_{k-1} is not adjacent with a pair of non-adjacent vertices of P_v . By assumption, w is adjacent with an internal vertex of $P_u[a, u]$ and an internal vertex of $P_v[v, b]$. Suppose $w = v_j$. So w is not adjacent with a vertex of $P_u[u_R, b]$ nor a vertex of $P_v[a, v_L]$.

Fact 1 *No vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$ and no vertex of $Q[v_{j+1}, v_k]$ is adjacent with a vertex of $P_u[u_R, b]$.*

Proof of Fact 1. Suppose some vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$. Let i be the largest integer less than j such that v_i is adjacent with a vertex of $P_v[a, v_L]$. Let z be a neighbour of v_i on $P_v[a, v_L]$ closest to v on this path. Then $C_1 : Q[v_i, v]P_v[v, z]v_i$ is a cycle of length at least 4. If $i \leq j-2$, then C_1 has length at least 5 and three consecutive vertices of C_1 are not incident with a chord of the cycle. This implies that G has a hole; which is forbidden. So $i = j-1$. Clearly $j \leq k-1$. Let $C_2 : P_v[z, v]Q[v, v_{j+1}]z$. Then C_2 is a cycle of length at least 3. Thus $\langle V(C_2) \rangle$ contains an induced cycle C' of length at least 3 that contains the edge zv_{j+1} . Since G contains no holes, C' has length 3 or 4. Since neither v_j nor v_{j-1} is adjacent with a vertex of $P_v[z, v] - z$ nor a vertex of $Q[v_{j+2}, v]$ and as $v_jz \notin E$, it is not difficult to see that the vertices of C_2 and C' induce a house or a domino. So no vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$. By an identical argument we can show that no vertex of $Q[v_{j+1}, v_k]$ is adjacent with a vertex of $P_u[u_R, b]$. \square

Fact 2 *No vertex of $P_v[a, v_L]$ is adjacent with any vertex of $P_u[u_R, b]$.*

Proof of Fact 2. Let z be the first vertex of $P_v[a, v_L]$ that is adjacent with some vertex of $P_u[u_R, b]$. Let y be a neighbour of z on $P_u[u_R, b]$ that is closest to b . Then the path $P : P_v[a, z]zyP_u[y, b]$ is an induced $a-b$ path. So $N[V(P)]$ is m^3 -convex and hence contains all induced $a-b$ paths of length at least 3. Since $\{a, b\} \cap \{u, v\} = \emptyset$, and since both $P_u[a, u]$ and $P_v[v, b]$ contain an internal vertex adjacent with w , both P_u and P_v have length at least 3. So $N[V(P)]$ contains all the vertices of P_u and P_v and hence u and v . So $N[V(P)]$ also contains Q . Thus every vertex of Q is adjacent with a vertex of $P_v[a, z]$ or with a vertex of $P_u[y, b]$. But by assumption w is adjacent with an internal vertex of both $P_u[a, u]$ and $P_v[v, b]$. So w is adjacent with a pair of non-adjacent vertices of P_v or a pair of non-adjacent vertices of P_u , neither of which is possible. \square

From Facts 1 and 2, it follows that no vertex of the path $P_v[a, v]Q[v, v_{j-1}]$ is adjacent with a vertex of the path $Q[v_{j+1}, u]P_u[u, b]$. Hence the subgraph induced by the path $P_v[a, v]Q[v, u]P_u[u, b]$ is an induced $a-b$ path that contains w ; contrary to the assumption that $w \notin I_m[a, b]$. This completes the proof of Case 2.

Case 3 Suppose that u belongs to an induced $a-b$ path P_u and v to an induced $a-b$ path P_v where P_u and P_v intersect at vertices other than a and b . We may assume that u and v do not both belong to P_u nor both to P_v ; otherwise, Case 1 occurs. Let a' be the last vertex prior to u on $P_u[a, u]$ that is also a vertex of P_v (perhaps $a' = a$). Let b' be the first vertex after u on $P_u[u, b]$ that belongs to P_v .

So $a'b' \notin E$. Let a'' be the last vertex prior to v on $P_v[a, v]$ that also belongs to P_u and b'' the first vertex after v on $P_v[v, b]$ that also belongs to P_u . So $a''b'' \notin E$.

Subcase 3.1 Suppose $P_u[a'', b'']$ contains both a' and b' . (Note b'' may precede a'' on $P_u[a'', b'']$.) In this case we can apply the argument used in Case 2 with a and b replaced by a'' and b'' and P_u and P_v replaced by $P_u[a'', b'']$ and $P_v[a'', b'']$. Hence this subcase cannot occur.

Subcase 3.2 Suppose $P_u[a'', b'']$ does not contain both a' and b' . Then a'' and b'' either lie on $P_u[a, a']$ or on $P_u[b', b]$. We will assume the former case occurs. The arguments for the latter case are similar. We may assume a'' precedes b'' on $P_u[a, a']$. The case where b'' precedes a'' on $P_u[a, a']$ is similar. First suppose that $P_v[a'', b'']$ has length 2. Then v is the only interior vertex of $P_v[a'', b'']$ and v is adjacent with two nonadjacent vertices of P_u . Let x be the first vertex on P_u that is adjacent with v , and y the last vertex of P_u adjacent with v . Since $uv \notin E$, $y \neq u$. If y precedes u on P_u , then the path obtained by taking $P_u[a, x]$ followed by xvy and then $P_u[y, b]$ is an induced $a - b$ path that contains both u and v . Thus we can apply the argument used in Case 1 to this path to obtain a contradiction. If y follows u on P_u , then we can use the path $P_u[x, y]$ and the path xvy and apply the argument used in Case 2 with x and y instead of a and b , respectively.

We now assume that $P_v[a'', b'']$ has length at least 3. Since $H = P_u[a'', b''] \setminus \{a'', b''\}$ is connected it follows, from Theorem 2, that $N[V(H)]$ is m^3 -convex. Since $N[V(H)]$ contains both a'' and b'' it must contain every internal vertex of $P_v[a'', b'']$. So each internal vertex of $P_v[a'', b'']$ is adjacent with an internal vertex of $P_u[a'', b'']$. If no internal vertex of $P_v[a'', b'']$ is adjacent with a vertex of $P_u[a, a''] \setminus \{a''\}$ or $P_u[b'', b] \setminus \{b''\}$, then we can replace $P_u[a'', b'']$ in P_u with $P_v[a'', b'']$ to obtain an induced $a - b$ path that contains both u and v . By applying the argument used in Case 1 to this path we obtain a contradiction. Let b''_L and b''_R be the neighbours of b'' that precede and succeed b'' on P_u . Let x be the neighbour of b'' on $P_v[a'', b'']$.

Suppose first that some internal vertex t of $P_v[a'', b'']$ is adjacent with some vertex y of $P_u[b''_R, b]$. If $t \neq x$, then t is also adjacent with some internal vertex z of $P_u[a'', b'']$. So $t \neq v$; otherwise, v is adjacent with two nonadjacent vertices of P_u which leads to a situation where the arguments of either Case 1 or Case 2 apply. If $P_u[z, y]$ has length at least 3, then it follows, from Theorem 2, that t is adjacent with every vertex of $P_u[z, y]$ including b'' ; this is not possible as t and b'' are nonadjacent vertices on the induced path $P_u[a'', b'']$. So $z = b''_L$ and $y = b''_R$ and b''_L is the only vertex of $P_u[a'', b'']$ to which t is adjacent. Suppose $P_v[t, b'']$ contains v . If $P_v[t, b'']$ contains at least four vertices, then the subgraph induced by b''_R and the vertices of $P_v[t, b'']$ must contain a hole, house or domino. (We use the fact that v cannot be adjacent to nonadjacent vertices of P_u ; otherwise, one can again argue that Case 1 or Case 2 occurs.) Suppose now that $P_v[t, b''] = tvb''$. Let d be the neighbour of b''_R on $P_u[b''_R, b]$. Then $\langle t, v, b'', b''_L, b''_R, d \rangle$ is a tailed twin C_4 since v is nonadjacent with b''_R and d .

Suppose thus that v does not belong to $P_v[t, b'']$. Then we may assume that

t is the first internal vertex on $P_v[a'', x]$ that is adjacent with b''_R . Let s be the neighbour of t on $P_v[a'', t]$. By the above we know that $tb''_L \in E$. If $sb''_L \in E$, then $\langle \{s, t, b''_L, b'', b''_R\} \rangle$ is a house which is forbidden. So assume $sb''_L \notin E$. Let c be the neighbour of b''_L on $P_u[a'', b''_L]$. Since $tc \notin E$ and G has no holes, $sc \in E$. But then $\langle \{s, c, t, b''_L, b'', b''_R\} \rangle$ is a domino, which is forbidden. So x is the only internal vertex of $P_v[a'', b'']$ that is adjacent with vertices of $P_u[b''_R, b]$. Let y be the neighbour of a'' on $P_v[a'', b'']$ and let a''_L and a''_R be the neighbours of a'' on $P_u[a, a'']$ and $P_u[a'', b'']$, respectively. One can argue as in the previous situation that the only internal vertex of $P_v[a'', b'']$ that is possibly adjacent with a vertex of $P_u[a, a'']$ is y .

Now let y' be the first vertex on $P_u[a, a'']$ that is adjacent with y (possibly $y' = a''$) and let x' be the last vertex on $P_u[b'', b]$ to which x is adjacent (possibly $x' = b''$). If x' belongs to $P_u[b'', u]$, then the path obtained by taking $P_u[a, y']$ followed by $y'yP_v[y, x]$ and then $xx'P_u[x', b]$ is an induced $a - b$ path containing both u and v . By Case 1 this produces a contradiction. Suppose thus that x' belongs to $P_u[u, b] - u$. Then $P_u[y', x']$ and $y'yP_v[y, x]xx'$ are two internally disjoint $y' - x'$ paths containing u and v , respectively. By applying the arguments of Case 2 to these two paths we again obtain a contradiction. Hence Case 3 cannot occur either. \square

Lemma 2. *Suppose $G = (V, E)$ is a graph for which $(V, \mathcal{M}_{m^3}(G))$ is a convex geometry. Then for all $a, b \in V$, $I_{m^3}(I_m[a, b]) \subseteq I_m[a, b]$.*

Proof. By the above G is (house, hole, domino, \mathcal{R}_{C_4} , \mathcal{T}_{C_4})-free. If $ab \in E$, then $I_m[a, b] = \{a, b\} = I_{m^3}(\{a, b\}) = I_{m^3}(I_m[a, b])$. Suppose $ab \notin E$. So, by Lemma 1, $I_{m^3}(I_m[a, b]) \subseteq I_m[a, b]$ (in fact equality holds). If $I_{m^3}(I_m[a, b]) \subsetneq I_m[a, b]$, then there is a set $W = \{w_1, w_2, w_3\} \subseteq I_m[a, b]$ such that $I_{m^3}(W) \subsetneq I_m[a, b]$. So there is an minimal W -tree T that contains a vertex $x \notin I_m[a, b]$. Let $H = \langle V(T) \rangle$. Then x is a cut-vertex of H . Thus one of the vertices of W , say w_3 does not belong to the component of $H - x$ that contains w_1 nor the component containing w_2 . So x lies on an induced $w_3 - w_i$ path for $i = 1, 2$. Since, by Lemma 1, $I_m[a, b]$ is m^3 -convex it must be the case that x is adjacent with w_1, w_2 and w_3 ; otherwise, $x \in I_m[a, b]$. So x is on an induced path between every pair of nonadjacent vertices of W .

Case 1 Suppose two nonadjacent vertices of W lie on the same induced $a - b$ path P . Then x is adjacent with a pair of nonadjacent vertices of an induced $a - b$ path. Hence x lies on an induced $a - b$ path; contrary to assumption. So w_1, w_2 and w_3 cannot lie on the same induced $a - b$ path.

Case 2 Suppose that two adjacent vertices of W , say w_1 and w_2 , lie on an induced $a - b$ path P . By Case 1, w_3 does not lie on the same induced $a - b$ path as w_1 and w_2 . Let Q be an induced $a - b$ path containing w_3 . Let s_3 and t_3 be the neighbours of w_3 on $Q[a, w_3]$ and $Q[w_3, b]$, respectively. (Note that $w_3 \neq a$ or b ; otherwise, the vertices of W lie on the same induced $a - b$ path. So s_3 and t_3 are well-defined.) Since $w_1w_2 \in E$, $w_1w_3, w_2w_3 \notin E$. Hence $\{s_3, t_3\} \cap \{w_1, w_2\} = \emptyset$.

Since x cannot be adjacent with two nonadjacent vertices of Q , x cannot be adjacent with both s_3 and t_3 . We may assume $xt_3 \notin E$. The path $R : w_2xw_3t_3$ is a path of length 3 between two vertices of $I_m[a, b]$. By Lemma 1, $I_m[a, b]$ is m^3 -convex. If R is induced this would imply that $x \in I_m[a, b]$, contrary to assumption. Hence $w_2t_3 \in E$. Now $\langle \{w_1, w_2, x, w_3, t_3\} \rangle$ is a house unless $w_1t_3 \in E$.

If $xs_3 \notin E$, then we can argue as for t_3 that $s_3w_1, s_3w_2 \in E$. But then $\langle \{w_1, w_2, w_3, x, s_3, t_3\} \rangle$ is a replicated twin C_4 which is forbidden.

Suppose now that $xs_3 \in E$. Then $\langle \{s_3, w_3, t_3, w_2, x\} \rangle$ is a house unless $s_3w_2 \in E$. If $w_1s_3 \notin E$, the path $R : s_3xw_1t_3$ is an induced path, of length 3, between two vertices in $I_m[a, b]$ that contains x . Since $I_m[a, b]$ is m^3 -convex and R contains x this contradicts our assumption about x . So $w_1s_3 \in E$. However, then $\langle \{w_1, w_2, w_3, x, s_3, t_3\} \rangle$ is again a replicated twin C_4 which is forbidden. So this case cannot occur.

Case 3 Suppose that no two vertices of W lie on the same induced $a - b$ path in G . (We may also assume that $w_1w_3, w_2w_3 \notin E$.) Let P_i be an induced $a - b$ path containing w_i for $i = 1, 2, 3$. From the case we are in w_i is not equal to either a or b for $i = 1, 2, 3$. For $i = 1, 2, 3$, let s_i and t_i be the neighbours of w_i on $P_i[a, w_i]$ and $P_i[w_i, b]$, respectively.

Subcase 3.1 $\{s_1, t_1\} = \{s_2, t_2\} = \{s_3, t_3\}$. Since s_1 and t_1 are non-adjacent vertices of P_1 , x is adjacent with at most one of s_1 or t_1 . Hence $\langle \{w_1, w_2, w_3, s_1, t_1, x\} \rangle$ is a replicated twin C_4 which is forbidden. So $\{s_3, t_3\}$ is either not equal to $\{s_1, t_1\}$ or $\{s_2, t_2\}$; suppose the former.

Subcase 3.2 $\{s_1, t_1\} \cap \{s_3, t_3\} = \emptyset$. Since s_i and t_i are non-adjacent vertices of P_i , x cannot be adjacent with both s_i and t_i for $i = 1, 2, 3$. So we may assume $xt_1 \notin E$. Suppose first that $xt_3 \notin E$. Since $t_1w_1xw_3$ is a path of length 3 between two vertices of $I_m[a, b]$ that contains x , it follows from Lemma 1 that this is not an induced path. Hence $w_3t_1 \in E$. Similarly by considering the path $w_1xw_3t_3$ and using the same argument it follows that $w_1t_3 \in E$. Similarly by considering the paths $w_2xw_3t_1$ and $w_2xw_3t_3$, it follows that w_2t_1 and $w_2t_3 \in E$. But now $\langle \{w_1, w_2, w_3, t_1, t_2, x\} \rangle$ is a replicated twin C_4 which is forbidden. So this case cannot occur.

Subcase 3.3 $|\{s_1, t_1\} \cap \{s_3, t_3\}| = 1$. We may assume $s_1 \in \{s_3, t_3\}$. The case where $t_1 \in \{s_3, t_3\}$ can be argued similarly. Suppose first that $s_1 = s_3$. If $s_1x \in E$, then $xt_1, xt_3 \notin E$. But then we can argue similarly as in Subcase 3.2 that $\langle \{w_1, w_2, w_3, t_1, t_3, x\} \rangle$ is a replicated twin C_4 . Hence $s_1x \notin E$. Suppose at least one of xt_1 or xt_3 is in E , say $xt_1 \in E$. Then $\langle \{s_1, w_1, t_1, x, w_3\} \rangle$ is a house unless $t_1w_3 \in E$. By considering the path $w_2xw_3s_1$ we can argue as before that $w_2s_1 \in E$. By now considering the path $t_1xw_2s_1$ it follows that $t_1w_2 \in E$. Thus $\langle \{w_1, w_2, w_3, s_1, t_1, x\} \rangle$ is a replicated twin C_4 which is forbidden. If neither xt_1 nor xt_3 are in E , then one can argue in a similar manner that $\langle \{w_1, w_2, w_3, s_1, x, t_3\} \rangle$ is a replicated twin C_4 . If $s_1 = t_3$ we can

argue similarly that G contains a replicated twin C_4 which is forbidden. Hence this case cannot occur either. This completes the proof of the lemma. \square

Lemma 3. *If $G = (V, E)$ is a (house, hole, domino, \mathcal{T}_{C_4})-free graph that contains an induced A -graph as labeled in Fig.4, then $u_2 \notin I_m[a, b]$.*

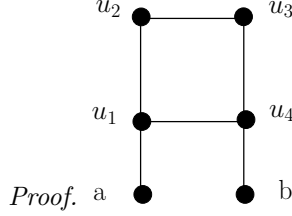


Figure 4: A labeled A -graph

Suppose, to the contrary, that $u_2 \in I_m[a, b]$ and let P be an induced $a - b$ path containing u_2 .

Case 1 $u_1 \notin V(P[a, u_2])$. Suppose $P[a, u_2] : aw_1w_2 \dots w_ku_2$. If $k = 1$, then $\langle \{a, w_1, u_2, u_1, u_4, u_3\} \rangle$ is a domino unless at least one of $w_1u_4, w_1u_3, w_1u_1 \in E$. If $w_1u_3 \notin E$, then w_1u_1 or $w_1u_4 \in E$. Suppose $w_1u_1 \in E$. Then $\langle \{w_1, u_2, u_3, u_1, u_4\} \rangle$ is a house unless $w_1u_4 \in E$. So in either case $w_1u_4 \in E$. But then $\langle \{u_2, w_1, u_1, u_3, u_4, b\} \rangle$ is a tailed twin C_4 which is forbidden. So $w_1u_3 \in E$. Since $\langle \{w_1, a, u_1, u_4, u_3\} \rangle$ is not a hole, either w_1u_1 or w_1u_4 is in E . If $w_1u_4 \notin E$, then $\langle \{w_1, a, u_1, u_4, u_3\} \rangle$ is a house which is forbidden. Hence $w_1u_4 \in E$. So if $P[a, u_2]$ has length 2, then its interior vertex is adjacent with both u_3 and u_4 .

Suppose now that $k \geq 2$. By Theorem 2, $N[u_1]$ is m^3 -convex. Since $N[u_1]$ contains both a and u_2 , every vertex of $P[a, u_2]$ is adjacent with u_1 . However, then $\langle \{w_k, u_1, u_2, u_3, u_4\} \rangle$ is a house unless w_ku_3 or w_ku_4 is in E . If $w_ku_3 \notin E$, then $w_ku_4 \in E$ and so $\langle \{u_4, u_1, u_3, w_k, u_2, b\} \rangle$ is a tailed twin C_4 which is forbidden. If $w_ku_3 \in E$ and $w_ku_4 \notin E$, then $\langle \{u_1, w_k, u_2, u_3, u_4, a\} \rangle$ is a tailed twin C_4 which is forbidden. Hence $w_ku_3, w_ku_4 \in E$.

Thus neither u_3 nor u_4 belongs to $P[u_2, b]$.

Suppose first that $P[u_2, b]$ has length 2. Let v_1 be its interior vertex. By Theorem 2, $N[v_1]$ is m^3 -convex. Since $N[v_1]$ contains both u_2 and b , v_1 is adjacent with every vertex on every induced $u_2 - b$ path of length at least 3. So v_1 is adjacent with u_3 and u_4 . But now $\langle \{w_k, u_2, v_1, u_4, b\} \rangle$ is a house which is forbidden.

Suppose now that $P[u_2, b]$ has length at least 3, say $P[u_2, b] : u_2v_1v_2 \dots v_rb$. By Theorem 2, $N[\{u_3, u_4\}]$ is m^3 -convex. Since $u_2, b \in N[\{u_3, u_4\}]$, every vertex of $P[u_2, b]$ is adjacent with either u_3 or u_4 . Let $b = v_{r+1}$. Let i be the smallest integer such that $v_iu_4 \in E$, possibly $i = r + 1$. Then $w_ku_2v_1 \dots v_iu_4w_k$ is an induced cycle. Since G has no holes $i = 1$. Let j be the smallest integer greater than 1 such that $v_ju_4 \in E$; possibly $j = r + 1$. If $j = 2$, then $\langle \{w_k, u_2, v_1, v_2, u_4\} \rangle$ is a house which is forbidden. Thus $j = 3$; otherwise,

$u_4v_1v_2 \dots v_ju_4$ is an induced cycle of length at least 5; which is forbidden. But then $\langle \{w_k, u_2, v_1, v_2, v_3, u_4\} \rangle$ is a domino which is again forbidden.

Case 2 $u_1 \in V(P[a, u_2])$. By considering $P[u_2, b]$ one can argue as in the previous case that G contains a forbidden subgraph. Hence the lemma follows. \square

We now complete the proof of the theorem. By the above G is (house, hole, domino, \mathcal{R}_{C_4} , \mathcal{T}_{C_4})-free. Suppose G contains the A graph as an induced subgraph. Then the collection of extreme vertices for the convex hull, $CH(A)$, of the A graph is a subset of the set of two leaves of the A graph. By Lemma 3 the monophonic interval of the leaves of the A graph does not include all the vertices of the A -graph. By Lemmas 1 and 2, $I_m[a, b]$ is m_3^3 -convex for all $a, b \in V$. This is true in particular for the two leaves of the A graph. Hence the convex hull of the extreme vertices of $CH(A)$ is thus not equal to $CH(A)$. This contradicts the fact that $(V, \mathcal{M}_{m_3^3}(G))$ is a convex geometry. \square

References

- [1] H.-J. Bandelt and H.M. Mulder, Distance-hereditary graphs. *J. Combin. Theory B* **41** (1986) 182–208.
- [2] J.M. Bilbao and P.H. Edelman, The Shapley value on convex geometries. *Discr. Appl. Math.* **103** (2000) 33–40.
- [3] A. Brandstädt, V.B. Le and J. P. Spinrad, *Graph Classes: A survey*. SIAM Monograph on Discrete Mathematics and Applications, Philadelphia (1999).
- [4] J. Cáceres and O.R. Oellermann, On 3-Steiner simplicial elimination. *Discr. Math.* **309** (2009) 5825–5833.
- [5] F.F. Dragan, F. Nicolai and A. Brandstädt, Convexity and HHD -free graphs. *SIAM J. Discr. Math.* **12** (1999) 119–135.
- [6] M. Farber and R.E. Jamison, Convexity in graphs and hypergraphs. *SIAM J. Alg. Disc. Math.* **7** (1986) 433–444.
- [7] E. Howorka, A characterization of distance hereditary graphs. *Quart. J. Math. Oxford* **28** (1977) 417–420.
- [8] B. Jamison and S. Olariu, On the semi-perfect elimination. *Adv. Appl. Math.* **9** (1988) 364–376.
- [9] E. Kubicka, G. Kubicki and O.R. Oellermann, Steiner intervals in graphs. *Discr. Math.* **81** (1998) 181–190.
- [10] M. Nielsen and O.R. Oellermann, Steiner trees and convex geometries. *SIAM J. Discr. Math* **23** (2009) 680–693.

- [11] O.R. Oellermann, Convexity notions in graphs:
<http://www-ma2.upc.edu/seara/wmcgt06/>
- [12] O.R. Oellermann and M.L. Puertas, Steiner intervals and Steiner geodetic numbers in distance hereditary Graphs. *Discr. Math.* **307** (2007) 88–96.
- [13] M.J.L. Van de Vel, *Theory of convex structures*. North-Holland, Amsterdam (1993).